



Fixed Point Results in Cone Metric Spaces for Multivalued Maps

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ABSTRACT

The aim of this paper is to generalize some results which are obtained by Kikkawa and Suzuki (2008) and others to the setting of cone metric spaces.

Keywords: Cone metric, fixed point, multivalued map.

1. INTRODUCTION

Banach contraction principle is widely recognized as the source of metric fixed point theory. This principle plays an important role in several branches of mathematics. A multivalued version of the Banach contraction principle was obtained by Nadler (1976). He used the concept of Hausdorff metric which is defined by

$$H(A, B) = \max\{\sup_{y \in B} d(y, A), \sup_{x \in A} d(x, B)\}$$

for $A, B \in CB(X)$ and $d(x, B) = \inf_{y \in B} d(x, y)$.

Berinde-Berinde (2007) gave a generalization of Nadler's fixed point Theorem and proved the following theorem:

Theorem 1. Let (X, d) be a complete metric space and $T: X \rightarrow CB(X)$. Assume that there exist a function $\alpha: [0, \infty) \rightarrow [0, 1)$ and $\ell \geq 0$ satisfying $\limsup_{r \rightarrow t^+} \alpha(r) < 1$, for every $t \in [0, \infty)$, such that

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + \ell d(y, Tx)$$

for all $x, y \in X$. Then T has a fixed point in X .

After that, Kikkawa and Suzuki (2008) gave another generalization of Nadler's result which is different from Berinde-Berinde Theorem. .

Theorem 2. Define a strictly decreasing function η from $[0, 1)$ into $(1/2, 1]$ by $\eta(r) = 1/(1+r)$. Let (X, d) be a complete metric space and T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $\eta(r)d(x, Tx) \leq d(x, y)$ implies $H(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Recently, Huang and Zhang (2007) introduced a cone metric space which is a generalization of a metric space. They generalized Banach contraction principle for cone metric spaces. Since then, many authors (Han and Xu (2013); Kunze *et al.* (2012); Nashine *et al.* (2013); Rezapour and Hamlbarani (2008); Shaddad and Noorani (2013) and Shatanawi *et al.* (2012)) obtain fixed point theorems in cone metric spaces in many various directions. Especially, the authors (Cho and Bae (2011); Cho *et al.* (2012); Latif and Shaddad (2010); Lin *et al.* (2012); Wardowski (2009) and Włodarczyk and Plebaniak (2012)) proved fixed point theorems for multivalued maps in cone metric spaces.

In this article, we give a generalization of Theorem 1 and Theorem 2 to the case of cone metric spaces. Furthermore, we extend and generalize Theorem 2.1 of Cho *et al.* (2012), Theorem 2.1 of Pathak-Shahzad (2009), Theorem 4.2 of Kamran-Kiran (2011) and others.

Consistent with Huang and Zhang (2007), the following notions, definitions and results will be needed in the sequel.

Let E be a real Banach space and P be a subset of E . P is called a cone if and only if

1. P is closed, $P \neq \emptyset$, $P \neq \{0\}$;
2. for all $x, y \in P \Rightarrow \alpha x + \beta y \in P$, where $\alpha, \beta \in \mathbb{R}^+$;
3. $P \cap -P = \{0\}$.

For a given cone $P \subset E$, we define a partial ordering \preceq with respect to P by the following: for $x, y \in E$, we say that $x \preceq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P .

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$

$$0 \preceq x \preceq y \Rightarrow \|x\| \leq K \|y\|.$$

The least positive number K satisfying this is called the normal constant of P (Huang and Zhang (2007)).

In this paper, we always suppose that E is a real Banach space, P is a cone in E , and \preceq is a partial ordering with respect to P .

Definition 3.

Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies

- (d1) $0 \preceq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$
- (d2) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space.

Definition 4.

Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X . Then

- (1) $\{x_n\}$ converges to $x \in X$ whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x) \preceq c$ for all $n \geq N$; we denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$;

- (2) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_m, x_n) \ll c$ for all $n, m \geq N$;
- (3) (X, d) is said to be complete if every Cauchy sequence in X is convergent in X .

The following remark was obtained by Kadelburg *et al.* (2009) is often used (in particular when dealing with cone metric spaces in which the cone need not be normal):

Remark 5.

- (1) If $u \ll v$ and $v \ll w$, then $u \ll w$.
- (2) If $0 \ll u \ll c$ for each $c \in \text{int}P$, then $u = 0$.
- (3) If $u \ll v + c$ for each $c \in \text{int}P$, then $u \ll v$.
- (4) If $0 \ll x \ll y$ and $0 \leq a$, then $0 \ll ax \ll ay$.
- (5) If $0 \ll x_n \ll y_n$ for each $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} y_n = y$, then $0 \ll x \ll y$.
- (6) If $c \in \text{int}P, 0 \ll a_n$ and $a_n \rightarrow 0$, then there exists n_0 such that for all $n > n_0$ we have $na_n \ll c$.

Let (X, d) be a cone metric space. We denote 2^X as a collection of nonempty subsets of X , $B(X)$ as a collection of nonempty bounded subsets of X , $Cl(X)$ as a collection of nonempty closed subsets of X and $CB(X)$ as a collection of nonempty closed and bounded subsets of X . An element $x \in X$ is called a fixed point of a multivalued map $T: X \rightarrow 2^X$ if $x \in T(x)$. Denote $Fix(T) = \{x \in X: x \in T(x)\}$. For $T: X \rightarrow Cl(X)$, and $x \in X$ we denote $D(x, Tx) = \{d(x, z): z \in Tx\}$. According to Cho and Bae (2011), we denote $s(p) = \{q \in E: p \ll q\}$ for $p \in E$, and $s(a, B) = \cup_{b \in B} s(d(a, b))$ for $a \in X$ and $B \in 2^X$.

For $A, B \in B(X)$ we denote $s(A, B) = (\cap_{a \in A} s(a, B)) \cap (\cap_{b \in B} s(b, A))$.

In 2011, Cho and Bae (2011) generalized the Nadler’s result Nadler (1976) to the setting of cone metric space. Moreover, they gave a useful lemma which will be used to prove our results.

Lemma 6. (Cho and Bae (2011)). Let (X, d) be a cone metric space, and let P be a cone in Banach space E .

- (1) Let $p, q \in E$. If $p \preceq q$, then $s(q) \subset s(p)$.
- (2) Let $x \in X$ and $A \in 2^X$. If $0 \in s(x, A)$, then $x \in A$.
- (3) Let $q \in P$ and let $A, B \in B(X)$ and $a \in A$. If $q \in s(A, B)$, then $q \in s(a, B)$.

In 2012, Cho *et al.* (2012) defined sequentially lower semicontinuous as follow

Definition 7. Let (X, d) be a cone metric space, and let $A \in 2^X$. A function $h : X \rightarrow 2^P - \{\emptyset\}$ defined by $h(x) = s(x, A)$ is called sequentially lower semicontinuous if for any $c \in \text{int}P$ there exists $n_0 \in \mathbb{N}$ such that $h(x_n) \subset h(x) - c$ for all $n \geq n_0$, whenever $\lim_{n \rightarrow \infty} x_n = x$ for any sequence $\{x_n\} \subset X$ and $x \in X$.

2. NEW RESULTS

Theorem 8. Let (X, d) be a complete cone metric space and $T : X \rightarrow CB(X)$. Let η be a nonincreasing function from $[0, 1)$ into $(1/2, 1]$ defined by $\eta(r) = 1/(1+r)$. Assume that there exists $r \in [0, 1)$. Assume for any $x \in X$ there exist $y \in Tx$ and $u \in D(x, Tx)$ such that $\eta(r)u \preceq d(x, y)$ implies $rd(x, y) \in s(Tx, Ty)$. Then T has a fixed point in X .

Proof.

Let $x_0 \in X$ and $x_1 \in Tx_0$ then there exists $u_0 = d(x_0, x_1) \in D(x_0, Tx_0)$ such that

$$\eta(r)u_0 \preceq d(x_0, x_1)$$

then

$$rd(x_0, x_1) \in s(Tx_0, Tx_1)$$

by lemma 6 we have

$$rd(x_0, x_1) \in s(x_1, Tx_1).$$

By definition, we can take $x_2 \in Tx_1$ such that

$$rd(x_0, x_1) \in s(d(x_1, x_2)).$$

So

$$d(x_1, x_2) \preceq rd(x_0, x_1).$$

Now, we can take $u_1 = d(x_1, x_2) \in D(x_1, Tx_1)$ such that

$$\eta(r)u_1 \leq d(x_1, x_2)$$

then

$$d(x_1, x_2) \in s(Tx_1, Tx_2)$$

this implies

$$rd(x_1, x_2) \in s(x_2, Tx_2)$$

take $x_3 \in Tx_2$, then we have

$$rd(x_1, x_2) \in s(d(x_2, x_3)).$$

Thus

$$d(x_2, x_3) \leq rd(x_1, x_2).$$

By induction we get an iterative sequence $\{x_n\}_{n \geq 0}$ in X such that for $x_n \in X$ there exists $u_n = d(x_n, x_{n+1}) \in D(x_n, Tx_n)$ such that

$$\eta(r)u_n \leq d(x_n, x_{n+1})$$

implies

$$rd(x_n, x_{n+1}) \in s(Tx_n, Tx_{n+1})$$

by lemma 6 we have

$$rd(x_n, x_{n+1}) \in s(x_{n+1}, Tx_{n+1})$$

By definition, we can take $x_{n+2} \in Tx_{n+1}$, then we have

$$rd(x_n, x_{n+1}) \in s(d(x_{n+1}, x_{n+2})).$$

Thus

$$d(x_{n+1}, x_{n+2}) \leq rd(x_n, x_{n+1}). \tag{1}$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then T has a fixed point. We assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Now, from (1) we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq rd(x_{n-1}, x_n) \\ &\leq r^2d(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq r^nd(x_0, x_1). \end{aligned}$$

Now, for $n > m$

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{n-1} r^i d(x_0, x_1) \\ &\leq \frac{r^m}{1-r} d(x_0, x_1) \end{aligned}$$

Since $r^m \rightarrow 0$ as $m \rightarrow \infty$, we obtain that $r^m/(1-r)d(x_0, x_1) \rightarrow 0$. Now, according to Remark 5 (6) and (1) we conclude that for $0 << c$ there is a natural number N_0 such that $d(x_m, x_n) << c$, for all $n, m \geq N_0$. So $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in (X, d) . Thus, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Now, we want to show that $x^* \in Tx^*$. First, we will prove that $\eta(r)u_n \leq d(x_{n+1}, x^*)$ for each $n \in \mathbb{N}$. For $c \in \text{int}P$ choose a natural number N_1 such that $d(x_n, x^*) << c$ and $rd(x_{n+1}, x^*) \in P$ for $n \geq N_1$. Thus, $c + rd(x_{n+1}, x^*) - d(x_n, x^*) \in P$, i.e., $d(x_n, x^*) \leq c + rd(x_{n+1}, x^*)$. By Remark 5 (3), we obtain $d(x_n, x^*) \leq rd(x_{n+1}, x^*)$ for $n \geq N_1$. As $x_{n+1} \in Tx_n$ we can take $u_n = d(x_n, x_{n+1}) \in D(x_n, Tx_n)$. We have

$$\begin{aligned} u_n = d(x_n, x_{n+1}) &\leq d(x_n, x^*) + d(x^*, x_{n+1}) \\ &\leq rd(x_{n+1}, x^*) + d(x^*, x_{n+1}) \\ &= (1+r)d(x^*, x_{n+1}). \end{aligned}$$

Thus

$$\eta(r)u_n \leq d(x_{n+1}, x^*).$$

Then

$$rd(x_{n+1}, x^*) \in s(Tx_{n+1}, Tx^*)$$

by Lemma 6 we have

$$rd(x_{n+1}, x^*) \in s(x_{n+2}, Tx^*)$$

where $x_{n+2} \in Tx_{n+1}$. Hence, by definition we can take $z_n \in Tx^*$ such that

$$d(x_{n+2}, z_n) \leq rd(x_{n+1}, x^*).$$

Now, for a given $c \in \text{int}P$ we choose a natural number $N_2 = \max\{N_0, N_1\}$ such that $d(x_n, x^*) \ll c/(r+1)$ for all $n \geq N_2$. Hence, for $n \geq N_2$ we have

$$\begin{aligned} d(x^*, z_n) &\leq d(x^*, x_{n+2}) + d(x_{n+2}, z_n) \\ &\leq d(x^*, x_{n+2}) + rd(x_{n+1}, x^*) \\ &\ll \frac{c}{r+1} + \frac{rc}{r+1} = c. \end{aligned}$$

Thus, $z_n \rightarrow x^*$. Since Tx^* is closed, $x^* \in Tx^*$.

Remark 9.

Theorem 8 is a generalization of Theorem 2 of Kikkawa and Suzuk (2008) from metric space to cone metric space without using normality of P . Moreover, we use the notion $s(Tx, Ty)$ which analogue the concept $H(Tx, Ty)$ in metric space.

The following example illustrates Theorem 8.

Example 10.

Let $X = [0,1]$, $E = C[0,1]$ and $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. Let $d: X \times X \rightarrow E$ be of the form

$$d(x, y) = \begin{cases} (x + y)e^t & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

and let $T: X \rightarrow Cl(X)$ defined by $Tx = [0, x/2]$. If we take $r = 2/3$ then $\eta(r) = 3/5$.

For $x = y$ it is trivial and for $x \neq y$ we can take $y = x/2 \in Tx$ and $u = d(x, x/3) \in D(x, Tx)$ for any $x \in X$. So $u = 4/3 xe^t$ and then $\eta(r)u = 4/5 xe^t \leq 3/2 xe^t = d(x, y)$. Now, we can choose $x/3 \in Tx$ and $x/4 \in Ty$ which satisfied $d(x/3, x/4) = 7/12 xe^t \leq xe^t = rd(x, y)$.

Thus, $rd(x, y) \in s(d(x/3, x/4)) \subset s(Tx, Ty)$. Hence T has a fixed point.

Theorem 11. Let (X, d) be a complete cone metric space and $T: X \rightarrow CB(X)$. Assume that there exist functions $\phi, \psi: P \rightarrow [0, 1)$ and $\ell \in \mathbb{R}^+$ satisfy the following

- (i) $\phi(t) + \psi(t) < 1$ for each $t \in P$ and $\limsup_{n \rightarrow \infty} \phi(r_n) + \psi(r_n) < 1$, for any decreasing sequence $\{r_n\} \in P$.
- (ii) for any $x, y \in X$, $\phi(d(x, y))d(x, y) + \psi(d(x, y))s(x, Tx) + \ell s(y, Tx) \subset ks(Tx, Ty)$

where $k \geq 1$. Then T has a fixed point in X .

Proof.

Let $x_0 \in X$ and $x_1 \in Tx_0$, then

$$\phi(d(x_0, x_1))d(x_0, x_1) + \psi(d(x_0, x_1))s(x_0, Tx_0) + \ell s(x_1, Tx_0) \subset ks(Tx_0, Tx_1).$$

Thus

$$\begin{aligned} &\phi(d(x_0, x_1))d(x_0, x_1) + \psi(d(x_0, x_1))s(d(x_0, x_1)) \\ &\quad + \ell s(d(x_1, x_1)) \subset ks(Tx_0, Tx_1). \end{aligned}$$

Then

$$\begin{aligned} &\phi(d(x_0, x_1))d(x_0, x_1) + \psi(d(x_0, x_1))d(x_0, x_1) + \ell d(x_1, x_1) \\ &\quad \in ks(Tx_0, Tx_1). \end{aligned}$$

By Lemma 6 we have

$$\phi(d(x_0, x_1))d(x_0, x_1) + \psi(d(x_0, x_1))d(x_0, x_1) \in ks(x_1, Tx_1).$$

Let us take $x_2 \in Tx_1$ then we have

$$\phi(d(x_0, x_1))d(x_0, x_1) + \psi(d(x_0, x_1))d(x_0, x_1) \in ks(d(x_1, x_2)).$$

Hence,

$$kd(x_1, x_2) \leq (\phi(d(x_0, x_1)) + \psi(d(x_0, x_1)))d(x_0, x_1).$$

Now, by using x_1, x_2 in condition (ii) we have

$$\begin{aligned} \phi(d(x_1, x_2))d(x_1, x_2) + \psi(d(x_1, x_2))s(x_1, Tx_1) + \ell s(x_2, Tx_1) \\ \subset ks(Tx_1, Tx_2). \end{aligned}$$

Since $x_2 \in Tx_1$ we obtain

$$\begin{aligned} \phi(d(x_1, x_2))d(x_1, x_2) + \psi(d(x_1, x_2))s(d(x_1, x_2)) + \ell s(d(x_2, x_2)) \\ \subset ks(Tx_1, Tx_2). \end{aligned}$$

Then

$$\begin{aligned} \phi(d(x_1, x_2))d(x_1, x_2) + \psi(d(x_1, x_2))d(x_1, x_2) + \ell d(x_2, x_2) \\ \in ks(Tx_1, Tx_2). \end{aligned}$$

By Lemma 6 we have

$$\phi(d(x_1, x_2))d(x_1, x_2) + \psi(d(x_1, x_2))d(x_1, x_2) \in ks(x_2, Tx_2)$$

taking $x_3 \in Tx_2$ we have

$$\phi(d(x_1, x_2))d(x_1, x_2) + \psi(d(x_1, x_2))d(x_1, x_2) \in ks(d(x_2, x_3)).$$

Therefore,

$$kd(x_2, x_3) \leq (\phi(d(x_1, x_2)) + \psi(d(x_1, x_2)))d(x_1, x_2).$$

By induction, we can construct a sequence $\{x_n\}_{n \geq 0}$ in X such that

$$kd(x_n, x_{n+1}) \leq (\phi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n)))d(x_{n-1}, x_n)$$

where $x_{n+1} \in Tx_n$. Thus,

$$d(x_n, x_{n+1}) \leq \frac{\phi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n))}{k} d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n) \tag{2}$$

We suppose that $x_{n+1} \neq x_n$ for each $n \geq 0$ because if $x_{n+1} = x_n$ for some n , then T has a fixed point.

From (2) $\{d(x_n, x_{n+1})\}_{n \geq 0}$ is a decreasing sequence in P . By (i) there exists $b \in (0,1)$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\phi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n)) < b.$$

Now,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{k} (\phi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n))) d(x_{n-1}, x_n) \\ &\leq \frac{1}{k^2} (\phi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n))) \\ &\quad (\phi(d(x_{n-2}, x_{n-1})) + \psi(d(x_{n-2}, x_{n-1}))) d(x_{n-2}, x_{n-1}) \\ &\leq \frac{1}{k^{n-n_0}} \prod_{i=n_0}^{n-1} (\phi(d(x_i, x_{i+1})) + \psi(d(x_i, x_{i+1}))) d(x_{n_0}, x_{n_0+1}) \\ &\leq \left(\frac{b}{k}\right)^{n-n_0} d(x_{n_0}, x_{n_0+1}). \end{aligned}$$

For $n > m \geq n_0$

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{n-1} \frac{b}{k^{i-n_0}} d(x_{n_0}, x_{n_0+1}) \\ &\leq \frac{b^{m-n_0}}{k^{m-n_0-1}(k-b)} d(x_{n_0}, x_{n_0+1}). \end{aligned}$$

Since $b^{m-n_0}/k^{m-n_0-1} \rightarrow 0$ as $m \rightarrow \infty$, we obtain that $(b^{m-n_0}/(k^{m-n_0-1})(k-b))d(x_{n_0}, x_{n_0+1}) \rightarrow 0$. Now, according to Remark 5 (6) and (1) we conclude that for $0 \ll c$ there is a natural number N_1 such that $d(x_m, x_n) \ll c$, for all $n, m \geq N_1$. So $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in (X, d) . Thus, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Now, we want to show that $x^* \in Tx^*$. We will create a sequence $\{z_n\}_{n \geq 0}$ in Tx^* such that, $z_n \rightarrow x^*$. From (ii) if we take $x = x_n, y = x^*$ we get

$$\begin{aligned} & \phi(d(x_n, x^*))d(x_n, x^*) + \psi(d(x_n, x^*))s(x_n, Tx_n) + \ell s(x^*, Tx_n) \\ & \subset ks(Tx_n, Tx^*). \end{aligned}$$

Then

$$\begin{aligned} & \phi(d(x_n, x^*))d(x_n, x^*) + \psi(d(x_n, x^*))s(d(x_n, x_{n+1})) + \ell s(d(x^*, x_{n+1})) \\ & \subset ks(Tx_n, Tx^*). \end{aligned}$$

So

$$\begin{aligned} & \phi(d(x_n, x^*))d(x_n, x^*) + \psi(d(x_n, x^*))d(x_n, x_{n+1}) + \ell d(x^*, x_{n+1}) \\ & \in ks(Tx_n, Tx^*). \end{aligned}$$

By Lemma 6 we get that

$$\begin{aligned} & \phi(d(x_n, x^*))d(x_n, x^*) + \psi(d(x_n, x^*))d(x_n, x_{n+1}) + \ell d(x^*, x_{n+1}) \\ & \in ks(x_{n+1}, Tx^*). \end{aligned}$$

Now, we can take $z_n \in Tx^*$ such that

$$\begin{aligned} & \phi(d(x_n, x^*))d(x_n, x^*) + \psi(d(x_n, x^*))d(x_n, x_{n+1}) + \ell d(x^*, x_{n+1}) \\ & \in ks(d(x_{n+1}, z_n)). \end{aligned}$$

Hence,

$$\begin{aligned} kd(x_{n+1}, z_n) & \leq \phi(d(x_n, x^*))d(x_n, x^*) + \psi(d(x_n, x^*))d(x_n, x_{n+1}) \\ & \quad + \ell d(x^*, x_{n+1}). \end{aligned}$$

Then

$$d(x_{n+1}, z_n) \leq \frac{\phi(d(x_n, x^*))}{k} d(x_n, x^*) + \frac{\psi(d(x_n, x^*))}{k} d(x_n, x_{n+1})$$

Now,

$$\begin{aligned} d(x^*, z_n) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, z_n) \\ &\leq d(x^*, x_{n+1}) + \frac{\phi(d(x_n, x^*))}{k} d(x_n, x^*) \\ &\quad + \frac{\psi(d(x_n, x^*))}{k} d(x_n, x_{n+1}) + \frac{\ell}{k} d(x^*, x_{n+1}) \\ &\leq d(x^*, x_{n+1}) + \frac{1}{k} d(x_n, x^*) + \frac{1}{k} d(x_n, x_{n+1}) + \frac{\ell}{k} d(x^*, x_{n+1}) \\ &\leq \frac{1}{k} (kd(x^*, x_{n+1}) + 2d(x_n, x^*) + d(x^*, x_{n+1}) + \ell d(x^*, x_{n+1})). \end{aligned}$$

Furthermore, for a given $c \in \text{int } P$ we choose a natural number N_2 such that $d(x_n, x^*) \ll kc/(3+k+\ell)$ for all $n \geq N_2$. Hence, for $n \geq N_2$ we have

$$d(x^*, z_n) \ll \frac{1}{k} \left(\frac{k^2 c}{3+k+\ell} + \frac{3kc}{3+k+\ell} + \frac{\ell kc}{3+k+\ell} \right) = c.$$

Thus, we get that $z_n \rightarrow x^*$. As Tx^* is closed, then $x^* \in Tx^*$.

If $k = 1$, $\ell = 0$ and a function $\psi(t) = 0$ for any $t \in P$, we have the following corollary which is a generalization of Mizoguchi-Takahashi's (1989) fixed point theorem.

Corollary 12. (Cho-Bae (2011)). Let (X, d) be a complete cone metric space and $T: X \rightarrow CB(X)$. Assume that there exists a function $\phi: P \rightarrow [0,1)$ satisfy the following

- (i) $\limsup_{n \rightarrow \infty} \phi(r_n) < 1$, for any decreasing sequence $\{r_n\} \in P$.
- (ii) for any $x, y \in X$, $\phi(d(x, y)) d(x, y) \in s(Tx, Ty)$

Then T has a fixed point in X .

If $k = 1$, $\psi(t) = 0$ for any $t \in P$, we have the following corollary which is a generalization of Theorem 4 of Berinde-Berinde (2007).

Corollary 13. Let (X, d) be a complete cone metric space and $T: X \rightarrow CB(X)$. Assume that there exist a function $\phi: P \rightarrow [0, 1)$ and $\ell \in \mathbb{R}^+$ satisfy the following

- (i) $\limsup_{n \rightarrow \infty} \phi(r_n) < 1$, for any decreasing sequence $\{r_n\} \in P$.
- (ii) for any $x, y \in X$, $\phi(d(x, y))d(x, y) + \ell s(y, Tx) \subset s(Tx, Ty)$

Then T has a fixed point in X .

If $k = 1$, $\phi(t)$ is a constant and $\psi(t) = 0$ for any $t \in P$ we have the following corollary which is a generalization of Theorem 3 of Berinde-Berinde (2007).

Corollary 14. Let (X, d) be a complete cone metric space, $T: X \rightarrow CB(X)$, $\alpha \in (0, 1)$ and $\ell \in \mathbb{R}^+$. Assume that for any $x, y \in X$, $\alpha d(x, y) + \ell s(y, Tx) \subset s(Tx, Ty)$

Then T has a fixed point in X .

Theorem 15. Let (X, d) be a complete cone metric space and $T: X \rightarrow Cl(X)$. Let $\theta: P \rightarrow E$ is a function with the following properties:

- (1) $\theta \geq 0$
- (2) $\theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2)$
- (3) θ is nondecreasing
- (4) $t \leq \theta(t)$.

Assume that there exists a function $\phi: P \rightarrow [0, k)$, $k < 1$ such that

- (i) $\limsup_{n \rightarrow \infty} \phi(r_n) < 1$, for any decreasing sequence $\{r_n\} \in P$.
- (ii) for every $x \in X$ there exists $y \in Tx$ such that $\phi(d(x, y))\theta(d(x, y)) \in s(\vartheta(D(y, Ty)))$

and

$$s\left(\vartheta(D(x, Tx))\right) \subset s\left(k\theta(d(x, y))\right)$$

where $\vartheta: 2^P - \{\emptyset\} \rightarrow 2^P - \{\emptyset\}$ defined by $\vartheta(D(x, Tx)) = \bigcup_{a \in Tx} \theta(d(x, a))$.

(iii) a function h is sequentially lower semicontinuous.

Then T has a fixed point in X .

Proof.

Let $x_0 \in X$ be arbitrary and fixed. There exists $x_1 \in Tx_0$ such that

$$\phi(d(x_0, x_1))\theta(d(x_0, x_1)) \in s\left(\vartheta(D(x_1, Tx_1))\right) \quad (3)$$

and

$$s\left(\vartheta(D(x_0, Tx_0))\right) \subset s\left(k\theta(d(x_0, x_1))\right)$$

Now, by using x_1 in condition (ii) there exists $x_2 \in Tx_1$ such that

$$\phi(d(x_1, x_2))\theta(d(x_1, x_2)) \in s\left(\vartheta(D(x_2, Tx_2))\right)$$

and

$$s\left(\vartheta(D(x_1, Tx_1))\right) \subset s\left(k\theta(d(x_1, x_2))\right) \quad (4)$$

From (3) and (4) we obtain

$$\phi(d(x_0, x_1))\theta(d(x_0, x_1)) \in s\left(k\theta(d(x_1, x_2))\right).$$

Thus

$$k\theta(d(x_1, x_2)) \preceq \phi(d(x_0, x_1))\theta(d(x_0, x_1)).$$

By continuing this process, we get a sequence $\{x_n\}_{n \geq 0}$ such that

$$k\theta(d(x_n, x_{n+1})) \preceq \phi(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n)).$$

Thus

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \frac{\phi(d(x_{n-1}, x_n))}{k} \theta(d(x_{n-1}, x_n)) \\ &\leq \theta(d(x_{n-1}, x_n)) \end{aligned} \tag{5}$$

Therefore, $\{\theta(d(x_n, x_{n+1}))\}_{n \geq 0}$ is a decreasing sequence in E . As θ is non decreasing, $\{d(x_n, x_{n+1})\}_{n \geq 0}$ is a decreasing sequence in P . By (i) there exists $b \in (0,1)$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\phi(d(x_{n-1}, x_n)) < b.$$

Now, from (5) we have

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \frac{1}{k} (\phi(d(x_{n-1}, x_n)) \theta(d(x_{n-1}, x_n))) \\ &\leq \frac{1}{k^2} \phi(d(x_{n-1}, x_n)) \phi(d(x_{n-2}, x_{n-1})) \theta(d(x_{n-2}, x_{n-1})) \\ &\leq \frac{1}{k^{n-n_0}} \prod_{i=n_0}^{n-1} \phi(d(x_i, x_{i+1})) \theta(d(x_{n_0}, x_{n_0+1})) \\ &\leq \left(\frac{b}{k}\right)^{n-n_0} \theta(d(x_{n_0}, x_{n_0+1})). \end{aligned}$$

For $n > m \geq n_0$ and by using (2) we obtain

$$\begin{aligned} \theta(d(x_m, x_n)) &\leq \theta\left(\sum_{i=m}^{n-1} d(x_i, x_{i+1})\right) \\ &\leq \sum_{i=m}^{n-1} \theta(d(x_i, x_{i+1})) \\ &\leq \sum_{i=m}^{n-1} \left(\frac{b}{k}\right)^{i-n_0} \theta(d(x_{n_0}, x_{n_0+1})) \\ &\leq \frac{b^{m-n_0}}{k^{m-n_0-1}(k-b)} \theta(d(x_{n_0}, x_{n_0+1})). \end{aligned}$$

Thus, for any $c \in \text{int } P$ there is a natural number N_1 such that $\theta(d(x_m, x_n)) \ll c$, for all $n, m \geq N_1$. We claim $\{x_n\}_{n \geq 0}$ is a Cauchy sequence, i.e., $d(x_m, x_n) \ll c$. Suppose not, then there exist subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ such that $c < d(x_{m_i}, x_{n_i}), \forall i$. Since θ is nondecreasing, then $\theta(c) < \theta(d(x_{m_i}, x_{n_i}))$. That is $\theta(c) < \theta(d(x_{m_i}, x_{n_i})) \ll c$, but from (4) we have $c \leq \theta(c)$. It is a contradiction. Hence, $\{x_n\}_{n \geq 0}$ is a Cauchy sequence. As (X, d) is complete, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Now, we want to show that $x^* \in Tx^*$. The function h is sequentially lower semicontinuous, so for any $c \in \text{int } P$, there exists $N_2 \in \mathbb{N}$ such that $s(x_n, Tx_n) \subset s(x^*, Tx^*) - c/2$ and $d(x_n, x_{n+1}) \ll c/2$ for each $n \geq N_2$. Since $s(x_n, Tx_n) \subset s(x^*, Tx^*) - c/2$, we obtain

$$s(d(x_n, x_{n+1})) \subset s(x^*, Tx^*) - \frac{c}{2}.$$

Then

$$d(x_n, x_{n+1}) \in s(x^*, Tx^*) - \frac{c}{2}.$$

Thus, we can take $z_n \in Tx^*$ such that

$$d(x_n, x_{n+1}) \in s(d(x^*, z_n)) - \frac{c}{2}.$$

Thus

$$d(x^*, z_n) - \frac{c}{2} \leq d(x_n, x_{n+1}).$$

By Remark 5 (1) we obtain that $d(x^*, z_n) - \frac{c}{2} \ll \frac{c}{2}$ which implies $d(x^*, z_n) \ll c$. Then $z_n \rightarrow x^*$. As Tx^* is closed, then $x^* \in Tx^*$, hence x is a fixed point of T .

Remark 16.

Theorem 15 is an extension of Theorem 2 of Pathak-Shahzad (2009) and Theorem 4 of Kamran-Kiran (2011) to cone metric space. Moreover, it is a generalization of Theorem 2 of Cho *et al.* (2012).

In Theorem 15 if we take the function $\theta = I$ identity function and the function $\phi = c$ constant, then we get the following result.

Corollary 17. (Cho *et al.* (2012)). Let (X, d) be a complete cone metric space and $T: X \rightarrow Cl(X)$ be a multivalued map. If there exist constants $c, k \in (0, 1]$ such that for any $x \in X$ there exists $y \in Tx$ such that

$$cd(x, y) \in s(y, Ty)$$

and

$$s(x, Tx) \subset s(kd(x, y))$$

then T has a fixed point in X provided $c < k$ and h is sequentially lower semicontinuous.

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